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ORIGINAL ARTICLE

Single and dual solutions of fractional order differential equations based on controlled Picard's method with Simpson rule

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Abstract This paper presents a semi-analytical method for solving fractional differential equations with strong terms like (exp, sin, cos,...). An auxiliary parameter is introduced into the well-known Picard's method and so called controlled Picard's method. The proposed approach is based on a combination of controlled Picard's method with Simpson rule. This approach can cover a wider range of integer and fractional orders differential equations due to the extra auxiliary parameter which enhances the convergence and is suitable for higher order differential equations. The proposed approach can be effectively applied to Bratu's problem in fractional order domain to predict and calculate all branches of problem solutions simultaneously. Also, it is tested on other fractional differential equations like nonlinear fractional order Sine-Gordon equation. The results demonstrate reliability, simplicity and efficiency of the approach developed.

1. Introduction

Fractional calculus and fractional differential equations (FDEs) have been considered as a source of many recent innovations during the last few decades, where the extra fractional-order parameters exhibit more flexibility to interpret many natural phenomena in different fields (Das and Pan, 2012; Monje et al., 2010; Petras, 2011; Podlubny, 1999; Semary et al. 2016).

Many approximations based on semi-analytical and numerical techniques were proposed to solve linear and nonlinear fractional- order differential equations that exist in many physical and engineering problems (Baskonus and Bulut, 2015; Baskonus and Bulut, 2016; Bulut et al. 2016; Chen et al., 2015, Diethelm et al., 2002; Gencoglu et al. 2017; Hashemi and Baleanu, 2016; Keshavarz et al. 2014; Parvizi et al., 2015). Picard's method introduced by Émile Picard in 1890, is a basic tool for proving the existence of solutions of initial value problems regarding ordinary first order differential

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equations. Recently, Picard's method was used to analyze and solve the integral and differential equations with different definitions of the derivative (Azarnavid et al., 2015; El-Sayed et al., 2014; Micula, 2015; Rontó et al. 2015; Salahshour et al., 2015; Vazquez-Leal et al., 2015). However, this method cannot provide us with a simple way to adjust and control the convergence region and the rate of giving an approximate series.

In this paper, we construct Picard's method with an auxiliary parameter h which proves very effective in controlling the convergence region of an approximate solution. Also, the combination between Picard's method with an auxiliary parameter and Simpson rule is proposed to solve nonlinear fractional differential equations in the form:

$$D^{\beta}u(t) + N[u(t)] = 0. \; u^{(i)}(0) = b_i, \quad i: 0(1)n-1, n-1 < \beta \leqslant n, \tag{1}$$

where $N[u(t)]$ contains strong nonlinear terms like (exp, sin, cos,...). The fractional order derivative (D^{β}) in Caputo sense defined by (Podlubny, 1999):

$$D^{\beta}u(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1}u^{(n)}(s)ds, \quad n-1<\beta<n, \tag{2}$$

and (J^{β}) is the Riemann–Liouville fractional integral operator of order $\beta \geqslant 0$ and defined by:

$$J^{\beta}u(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds. \tag{3}$$

The important property of the Caputo fractional derivative is:

$$J^{\beta}D^{\beta}u(t) = u(t) - \sum_{k=0}^{n-1} u^{(k)}(0)\frac{t^k}{k!}, \quad n-1<\beta\leqslant n. \tag{4}$$

2. The methodology

We apply the Riemann–Liouville integral of order β (J^{β}) on Eq. (1) and after making use of the property (4), we get the integrated form of Eq. (1), namely

$$u(t) = \sum_{k=0}^{n-1} u^{(k)}(0)\frac{t^k}{k!} - J^{\beta}N[u(t)], \tag{5}$$

where $u^{(k)}(0) = b_k, k = 0, 1, .., n-1$. Applying Picard's method to the integral Eq. (5), the solution can be reconstructed as follows:

$$u_{m+1}(t) = \sum_{k=0}^{n-1} b_k\frac{t^k}{k!} - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}N[u_m(s)]ds, \quad m \geqslant 0. \tag{6}$$

Adding and subtracting the term $\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}D^{\beta}u_m(s)ds$ in the right-hand side of (6), the iterative formula (6) becomes:

$$u_{m+1}(t) = \sum_{k=0}^{n-1} b_k\frac{t^k}{k!} - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\{D^{\beta}u_m(s) + N[u_m(s)]\}ds + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}D^{\beta}u_m(s)ds. \tag{7}$$

Using Caputo fractional order derivative (4), then Eq. (7) becomes:

$$u_{m+1}(t) = \sum_{k=0}^{n-1} b_k\frac{t^k}{k!} - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\{D^{\beta}u_m(s) + N[u_m(s)]\}ds + u_m(t) - \sum_{k=0}^{n-1} u_m^{(k)}\frac{t^k}{k!}. \tag{8}$$

The successive approximation $u_m(t)$ must satisfy the initial conditions, for that $u_m^{(k)} = b_k$ and the iterative formula (8) becomes:

$$u_{m+1}(t) = u_m(t) - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\{D^{\beta}u_m(s) + N[u_m(s)]\}ds. \tag{9}$$

The property (4) is right for the integer order case and one can prove it easily using integration by parts. So, it should be emphasized that the iteration formula (9) is suitable to solve the problem (1) for integer and fractional orders. The variational iteration method (VIM) is one of the famous techniques used to solve linear and nonlinear differential equations (Ghaneai and Hosseini, 2015; He, 1999; Wazwaz, 2009a; Semary and Hassan, 2015). To solve the integer order differential Eq. (1) by the variational iteration method (He, 1999; Wazwaz, 2009a), one can construct an iteration formula for the problem (1) as follows:

$$u_{m+1} = u_m + \int_0^t \lambda(s)(D^n u_m(s) + N[u_m(s)])ds, \tag{10}$$

where $\lambda(s)$ is a general Lagrange multiplier and it is equal $-\frac{(t-s)^{n-1}}{n-1!}$ (Wazwaz, 2009a).

Remark: *The Picard iterative formula (9) is the same variational iterative formula generated by the variational iteration method (10) when $\beta = n$ and the general Lagrange multiplier*

$$\lambda(s) = -\frac{(t-s)^{n-1}}{\Gamma(n)}.$$

2.1. Controlled Picard's method with Simpson rule

We consider the nonlinear fractional order differential Equation (1) in the form:

$$F[t, u(t), \beta] = D^{\beta}u(t) + N[u(t)] = 0. \tag{11}$$

Multiply h to both sides in Eq. (11) to become:

$$hF[t, u(t), \beta] = 0, \tag{12}$$

where h is an auxiliary parameter. Adding and subtracting $D^{\beta}u(t)$ from the left-hand side of (12) to become in the form:

$$D^{\beta}u(t) + hF[t, u(t), \beta] - D^{\beta}u(t) = 0, \tag{13}$$

and setting $N_1(t, u) = hF[t, u(t), \beta] - D^{\beta}u(t)$, the Eq. (13) is given by:

$$D^{\beta}u(t) + N_1(t, u) = 0. \tag{14}$$

Appling the Picard iteration Eq. (9) to the equation (14), we get:

$$u_{m+1}(t) = u_m(t) - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \{D^\beta u_m(s) + N_1(s, u_m)\} ds, \quad (15)$$

and replace $N_1(s, u_m)$ by $hF[s, u_m(t)] - D^\beta u_m(s)$ in Eq. (15), then the iteration formula with an auxiliary parameter for the problem (11) is given by:

$$u_{m+1}(t) = u_m(t) - \frac{h}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \{D^\beta u_m(s) + N(s, u_m)\} ds. \quad (16)$$

Using the important property (4), the iteration formula (16) becomes:

$$u_{m+1} = u_m - h\left(u_m - \sum_{k=0}^{n-1} u_m^{(k)}(0) \frac{t^k}{k!}\right) - hI(t), \quad (17)$$

where $I(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} N(s, u_m) ds$. If the term $N(s, u_m)$ contains powerful terms like (exp, sin, cos,...) it is difficult to obtain the value of $I(t)$ exactly. Therefore, we can approximate the value of $I(t)$ using composite Simpson rule (Atkinson, 1989). The rule is as follows:

$$I(t) = \int_0^t y(s) ds \approx \frac{t}{6n}\left\{y_0 + y_{2n} + \sum_{i=1}^{n} 4y_{2i-1} + \sum_{i=1}^{n-1} 2y_{2i}\right\}, \quad (18)$$

where $2n$ is the number of subintervals and $y(s) = \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} N(s, u_m)$, $y_0 = y(0), y_{2n} = y(t)$ and $y_i = y\left(\frac{it}{2n}\right), i : 1(1)2n-1$. Then the iterative formula (17) to solve the problem (1) becomes:

$$u_{m+1}(t,h) = u_m - h\left(u_m - \sum_{k=0}^{n-1} u_m^{(k)}(0) \frac{t^k}{k!}\right) - \frac{th}{6n}\left\{y_0 + y_{2n} + \sum_{i=1}^{n} 4y_{2i+1} + \sum_{i=1}^{n-1} 2y_{2i}\right\} \quad (19)$$

It should be emphasized that $u_{m+1}(t,h)$ can be computed by symbolic software programs such as Wolfram Mathematica or Maple, starting by an initial approximation $u_0(t,h) = \sum_{k=0}^{n-1} b_k \frac{t^k}{k!}$ or which satisfies at least the initial conditions for the problem. We obtain the approximate solution $u_{m+1}(t,h)$ for the problem (1) but there is still an unknown parameter in the approximate solution $u_{m+1}(t,h)$ the auxiliary parameter h, which should be determined. In general, by means of the so-called h-curve (Liao, 2003), it is straightforward to choose a proper value of h which ensures that the approximate solutions are convergent as follows. Let $d \in \Omega$, then $u(d,h), u'(d,h)$ are functions of h and the curves of these functions versus h results in a horizontal line segment which corresponds to the valid region of h.

3. Numerical examples

Example 1: Consider the following initial value problem:

$$D^\beta u(t) + e^t u^2(t) = t^8 e^t + \frac{\Gamma(5) t^{4-\beta}}{\Gamma(5-\beta)}, 2 < \beta \leqslant 3, u(0) = u'(0) = u''(0) = 0, \quad (20)$$

and has the exact solution $u(t) = t^4$.

Applying the iteration formula (19) to equation (20) by taking $n = 8$, we get

$$u_{m+1}(t,h) = (1-h)u_m(t,h) - \frac{th}{48}\left\{y_0 + y_{16} + \sum_{i=1}^{7} 2y_{2i} + \sum_{i=1}^{8} 4y_{2i-1}\right\}, \quad (21)$$

where $y_i = \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\{e^s u_m^2(s) - s^8 e^s - \frac{24 s^{4-\beta}}{\Gamma(5-\beta)}\}\Big|_{s=\frac{it}{16}}$ $\forall i : 0(1)16$. Starting by $u_0(t,h) = 0$ which satisfies the initial conditions (20). Using the Wolfram Mathematica 9 software, starting with $u_0(t,h)$. We can obtain the successive approximations $u_{m+1}(t,h)$, $m \geqslant 0$. For example, when $\beta = 2.5$, the first approximate solution $u_1(t,h)$ is given by:

$$\begin{aligned} u_1(t,h) = 1.0004ht\{t^{3.} &+ (1.32 \times 10^{-11} e^{0.0625t} + 1.528 \times 10^{-9} e^{0.125t} + 7.0104 \times 10^{-8} e^{0.1875t} + 3.105 \times 10^{-7} e^{0.25t} + 0.00000324 e^{0.3125t} \\ &+ 0.000006053 e^{0.375t} + 0.0000354 e^{0.4375t} + 0.0000432 e^{0.5t} + 0.000181 e^{0.5625t} \\ &+ 0.000167 e^{0.625t} + 0.000546 e^{0.6875t} + 0.000392 e^{0.75t} + 0.000966 e^{0.8125t} \\ &+ 0.000475 e^{0.875t} + 0.00058 e^{0.9375t}) t^{9.5}\}, \end{aligned} \quad (22)$$

and so on, to obtain a suitable value of an auxiliary parameter h. Fig. 1 shows h-curve for 5th-order approximation at different values of $t = d$. From this figure, it is clear that the valid region of h is [0.5, 1.5], whose line segment gives a constant value $u_5(d,h)$. According to Fig. 1, we select $h = 1 \in [0.5, 1.5]$. To show the accuracy of the present method solution, the absolute error of $u_5(t)$ for different values of β is given in Fig. 2.

Example 2: Consider Bratu's problem in fractional order domain as follows:

$$D^\alpha u(t) + \lambda e^u = 0, u(0) = u(1) = 0, \ 1 < \alpha \leqslant 2 \quad (23)$$

Bratu's problem is a nonlinear two boundary value problem with a strong nonlinear term e^u and parameter λ. The integer order problem appears in a number of applications such as the fuel ignition model of the thermal combustion theory. The model of thermal reaction process, the Chandrasekhar model of the expansion of the Universe, questions in geometry and

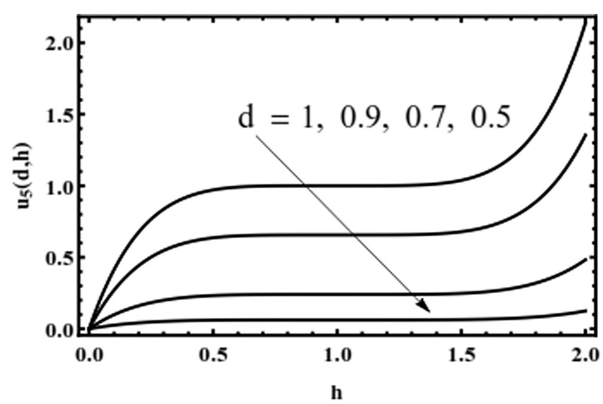

Figure 1 h-curve of $u_5(d,h)$ with $\beta = 2.5$ of example 1.

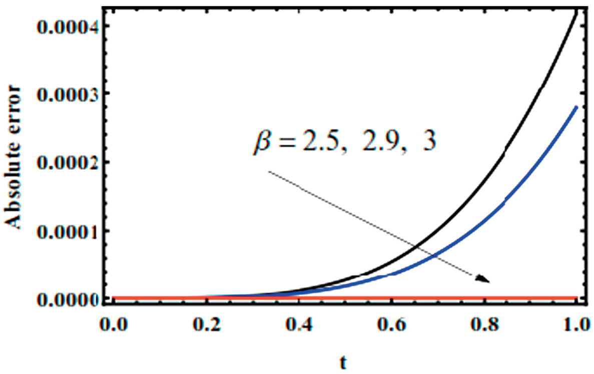

Figure 2 Absolute error of solution by proposed method of example 1.

relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology (Jalilian, 2010; Wazwaz, 2005; Wazwaz, 2012). For integer order case ($\alpha = 2$), the analytical solution of the problem (15) can be put in the following form (Wazwaz, 2005):

$$u(t) = -2\ln\frac{\cosh\left((t-0.5)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}, \tag{24}$$

where θ is the solution of the equation $\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$. The problem has no, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c given by $\lambda_c = 3.513830719$ and $u'(0) = \theta\tanh(\frac{\theta}{4})$ as shown in Fig. 3 when $\alpha = 2$. The integer order case was solved using the semi-analytic and numerical techniques (Hassan and El-Tawil, 2011; Hassan and Semary, 2013; Jalilian, 2010; Kafri and Khuri, 2016; Semary and Hassan, 2015) using spline method (Jalilian, 2010), variational iteration method (Semary and Hassan, 2015), homotopy analysis method (Hassan and El-Tawil, 2011; Hassan and Semary, 2013) and other methods.

The purpose of this paper is to solve and to show how one can find out the existence of dual solutions for the problem

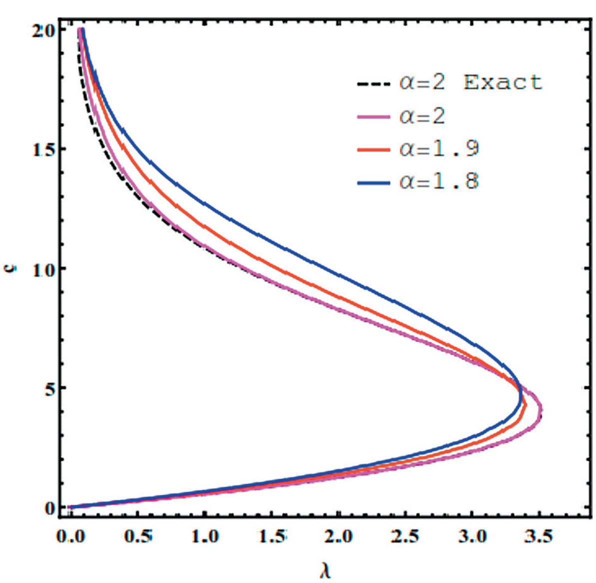

Figure 3 The multiplicity curve of the Bratu's problem (23).

(23) in fraction order domain. To apply the present method, suppose that $u'(0) = c$, so the problem becomes:

$$D^{\alpha}u(t) + \lambda e^{u} = 0, \tag{25a}$$

Subject to initial conditions

$$u(0) = 0, u'(0) = c, \tag{25b}$$

with additional forcing condition

$$u(1) = 0. \tag{26}$$

Now, we apply the present method on equation (25) by taking $n = 4$, thus the iterative formula (19) becomes:

$$u_{m+1}(t,c,h) = (1-h)u_m + hct - \frac{th}{24}\left\{y_0 + y_8 + \sum_{i=1}^{3}2y_{2i} + \sum_{i=1}^{4}4y_{2i-1}\right\}, \tag{27}$$

where $y_i = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\{\lambda e^{u_m(s,h)}\right\}\Big|_{s=\frac{it}{8}} \forall i : 0(1)8$. Using the initial solution $u_0(t) = ct - \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\lambda c t^{\alpha+1}}{\Gamma(\alpha+2)}$ which satisfies the initial condition (25b) which can be used to obtain the mth-successive approximations (27). The first approximation solution $u_1(t,c,h)$ is given by:

$$u_1(t,c,h) = \frac{1}{315\Gamma(\alpha)}2^{-1-3\alpha}f(t,c,h)ht^{\alpha}\lambda + \frac{(-1+h)t^{\alpha}\lambda\Gamma(2+\alpha) + ct\Gamma(1+\alpha)((-1+h)t^{\alpha}\lambda + \Gamma(2+\alpha))}{\Gamma(1+\alpha)\Gamma(2+\alpha)}, \tag{28a}$$

$$\begin{aligned} f(t,c,h) = &-1685^{\alpha}e^{-\frac{\left(\frac{3}{8}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[1+\alpha]}+\frac{3}{8}ct\left(1-\frac{\left(\frac{3}{8}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[2+\alpha]}\right)} \\ &- 2803^{\alpha}e^{-\frac{\left(\frac{5}{8}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[1+\alpha]}+\frac{5}{8}ct\left(1-\frac{\left(\frac{5}{8}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[2+\alpha]}\right)} \\ &- 1052^{1+\alpha}e^{-\frac{\left(\frac{3}{4}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[1+\alpha]}+\frac{3}{4}ct\left(1-\frac{\left(\frac{3}{4}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[2+\alpha]}\right)} \\ &- 840e^{-\frac{\left(\frac{7}{8}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[1+\alpha]}+\frac{7}{8}ct\left(1-\frac{\left(\frac{7}{8}\right)^{\alpha}t^{\alpha}\lambda}{\Gamma[2+\alpha]}\right)} \\ &- 1054^{\alpha}e^{-\frac{2^{-\alpha}t^{\alpha}\lambda}{\Gamma[1+\alpha]}+\frac{1}{2}ct\left(1-\frac{2^{-\alpha}t^{\alpha}\lambda}{\Gamma[2+\alpha]}\right)} \\ &- 352^{1+\alpha}3^{\alpha}e^{-\frac{4^{-\alpha}t^{\alpha}\lambda}{\lambda[1+\alpha]}+\frac{1}{4}ct\left(1-\frac{4^{-\alpha}t^{\alpha}\lambda}{\Gamma[2+\alpha]}\right)} \\ &- 1207^{\alpha}e^{\frac{1}{8}\left(ct-\frac{8^{-\alpha}t^{\alpha}\lambda(ct\Gamma[1+\alpha]+8\Gamma[2+\alpha])}{\Gamma[1+\alpha]\Gamma[2+\alpha]}\right)}, \end{aligned} \tag{28b}$$

and so on. Therefore the equation (26), with the help of forcing condition $u(1) = 0$, becomes

$$u(1) \cong u_{m+1}(1,c,h) = 0, \tag{29}$$

According to the above equation, in Fig. 4 the c as a function of auxiliary parameter h, has been plotted in the h range [0, 2] implicitly, for $\alpha = 1.9$ and different values of λ. Two c-plateaus (two line segments giving constant values of c) can be identified in this figure, this means that there are two solutions for each value of λ and the method is convergent when $h = 1$. In general, the multiplicity curves for different values of the fractional order α are shown in Fig. 3. It is clear from this figure the approximate solution when $\alpha = 2$ is fully consistent with the exact solution. Also, the problem in fractional order domain has no, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ $\lambda < \lambda_c$ respectively, where the critical value λ_c

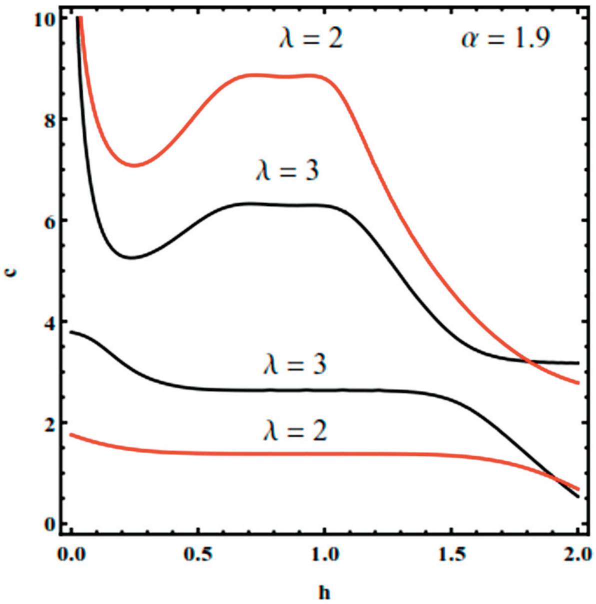

Figure 4 *h*-curve (29) with $m = 4$, $\alpha = 1.9$ when $\lambda = 2$ (red color) and $\lambda = 3$ (black).

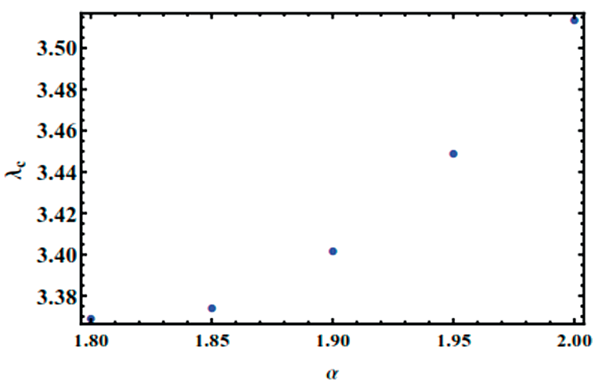

Figure 5 The value of critical λ against the fractional order α.

is shown in Fig. 5 and summarized in Table 1 for different values of α. For $\alpha = 2$, the exact values of $u'(0)$ from the analytic solution (24) are 2.319602 and 6.103 for $\lambda = 3$. By the present method, the approximate values of $u'(0)$ are 2.3197 and 6.105. To show the accuracy of these dual approximate solutions when $\alpha = 2$, the absolute errors for first and second solutions are shown in Figs. 6 and 7, respectively. Also, the approximate values of $u'(0)$ when $\alpha = 1.9$ and $\lambda = 3$ are 2.6325 and 6.285. In this case the problem solutions are shown in Fig. 8.

Table 1 The value of λ_c for different values of α.

α	λ_c	λ_c exact
2	3.5140	3.5138
1.95	3.4488	–
1.9	3.4015	
1.85	3.3739	
1.8	3.3693	

Example 3: Consider the nonlinear Sine-Gordon equation in fraction order domain as follows:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} - u_{xx} + \sin(u) = 0, 1 < \alpha \leqslant 2, \tag{30a}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = 4\mathrm{sech}(x) \tag{30b}$$

The sine-Gordon equation appears in a number of applications such as the propagation of fluxons in Josephson junc-

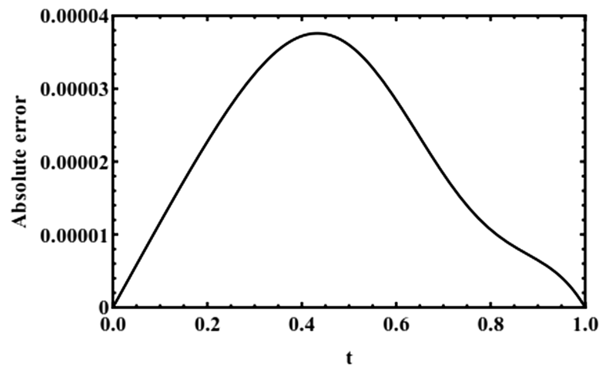

Figure 6 Absolute error for the first branch solution with $\alpha = 2$.

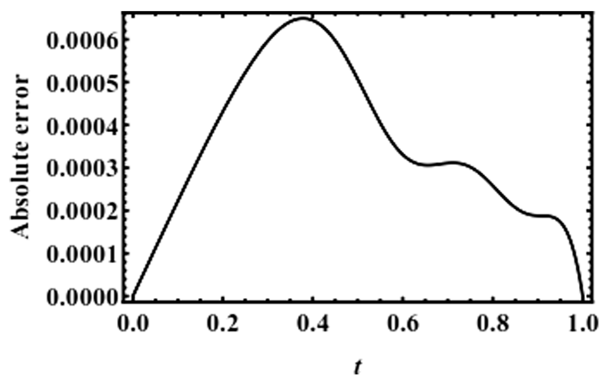

Figure 7 Absolute error for the second branch solution with $\alpha = 2$.

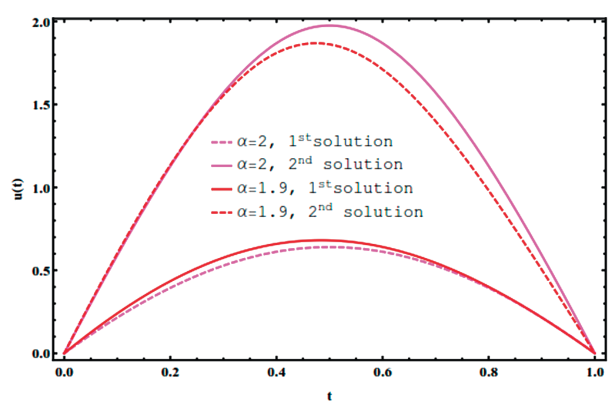

Figure 8 The proposed method solutions of Bratu's problem.

tions between two superconductors, nonlinear optics, solid state Physics, dislocations in crystals, stability of fluid motions, the motion of a rigid pendulum attached to a stretched wire and in the study of the differential geometry (Dodd et al. 1982; Wazwaz, 2009b; Whitham, 1999). For integer order case $\alpha = 2$, The exact solution is given by (Hassan and El-Tawil, 2012; Yousif and Mahmood, 2017):

$$u_E(x,t) = 4tan^{-1}(t \text{ sech } x) \tag{31}$$

Applying the iteration formula (19) to equation (30) by taking $n = 1$, we get

$$u_{m+1}(t,x,h) = (1-h)u_m + 4ht \text{ sech } x - \frac{th}{6}\{y_0 + 4y_1 + y_2\} \tag{32}$$

where $y_i = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\{-\frac{\partial^2}{\partial x^2}u_m(s,x) + \sin(u_m)\}\Big|_{s=\frac{it}{2}} \forall i: 0,1,2$. Using $u_0(t,x,h) = 4t \text{ sech } x$, then the first approximation solution is given by:

$$u_1(t,x,h) = 4t \text{ sech } x - \frac{1.33e^{-0.69\alpha}ht^{\alpha}(\sin(2t \text{ sech } x) - 2t(-\text{sech}^3 x + \text{sech} x \tanh^2 x))}{\Gamma(\alpha)} \tag{33}$$

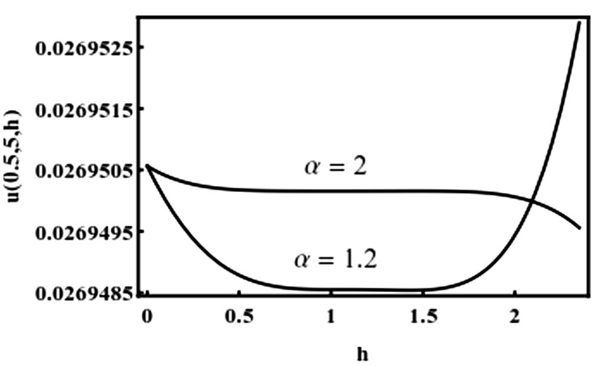

Figure 9 h-curve with 5th approximation of iteration formula (32).

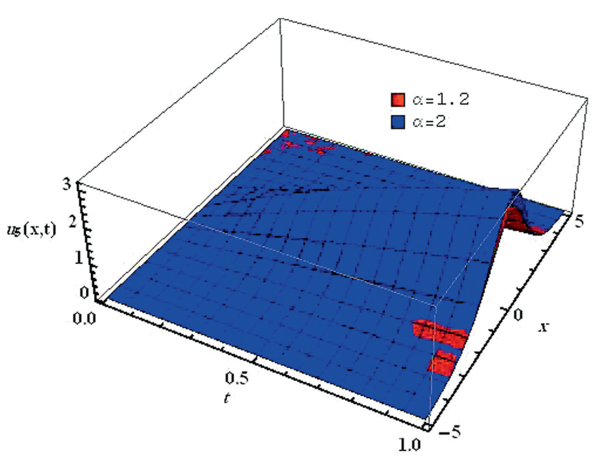

Figure 10 The Sin-Gordon problem solution when $\alpha = 2$ and 1.2.

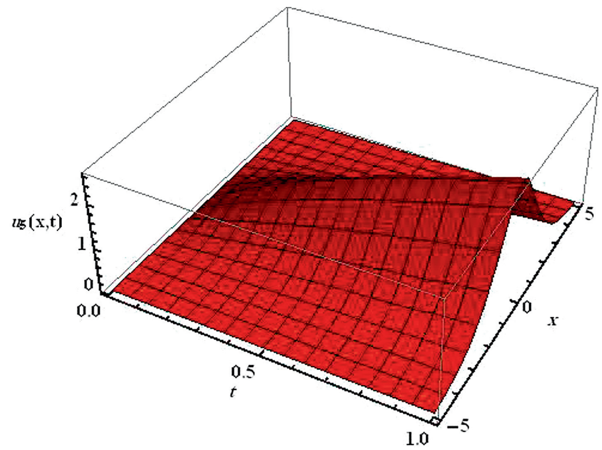

Figure 11 The Sin-Gordon problem solution with $\alpha = 1.2$.

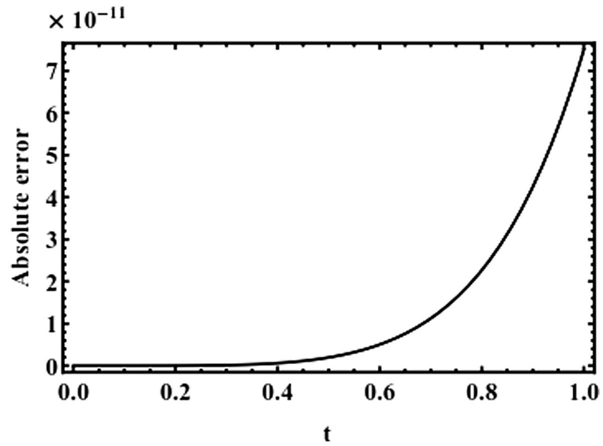

Figure 12 The absolute error when $\alpha = 2$ and $x = 5$ of example 3.

Fig. 9 shows a h-curve when $\alpha = 1.2$ and $\alpha = 2$. It is clear from this figure the method is convergent when $h = 1$. Figs. 10 and 11 show the solution for the problem when $\alpha = 1.2$ and $\alpha = 2$. Finally, Fig. 12 shows the absolute error when $\alpha = 2$.

4. Conclusions

In this paper, a controlled Picard's method is introduced based on the traditional Picard's method by adding an auxiliary parameter. By combining controlled Picard's method and the Simpson rule, a new computational method is presented for solving fractional differential equations with strongly nonlinear terms. The proposed approach provides a simple way to adjust and control the convergence region of approximate solution. The proposed approach succeeded in detecting dual solutions to Bratu's problem at the same time. The scheme is tested on three fractional order differential equations with different classes. The results demonstrate reliability and efficiency of the approach developed.

Conflict of interest

The authors have no conflict of interest.

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